Solutions: Homework 4

Nandagopal Ramachandran

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Problem 1. Show that $f(z) = |z|^2 = x^2 + y^2$ has a derivative only at the origin.

Proof. Suppose f is complex differentiable at $z = x + iy$ for some $x, y \in \mathbb{R}$. Then, for $h \in \mathbb{R}$, we have

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h \to 0} \frac{f(z+ih) - f(z)}{ih},$$

i.e.

$$\lim_{h \to 0} \frac{(x+h)^2 + y^2 - x^2 - y^2}{h} = \lim_{h \to 0} \frac{x^2 + (y+h)^2 - x^2 - y^2}{ih}$$

which reduces to $2x = -2iy$, and since $x, y \in \mathbb{R}$, we must have $x = y = 0$.
Now, at $z = 0$, for any $h \in \mathbb{C}$, $\frac{f(h)-f(0)}{h} = \frac{|h|^2}{h} = \overline{h}$, which converges to 0 as $h \to 0$. So f is complex differentiable at 0.

□

Problem 2. Let G and Ω be open in $\mathbb{C}, G$ connected and suppose f and h are functions defined on $G, g : \Omega \to \mathbb{C}$, and suppose that $f(G) \subset \Omega$. Suppose that g and h are analytic with $g'(z) \neq 0$ for all $z \in f(G)$, that f is continuous, h is one-one, and that they satisfy $h(z) = g(f(z))$ for z in G. Show that f is analytic. Give a formula for $f'(z)$.

Proof. Fix $a \in G$ and let $\epsilon \in \mathbb{C}$ be such that $\epsilon \neq 0$ and $a + \epsilon \in G$. Hence $h(a) = g(f(a))$ and $h(a+\epsilon) = g(f(a+\epsilon))$. Since h is one-one, we know that $h(a) \neq h(a+\epsilon)$, hence $f(a) \neq f(a+\epsilon)$. Also

$$\frac{h(a+\epsilon) - h(a)}{\epsilon} = \frac{g(f(a+\epsilon)) - g(f(a))}{f(a+\epsilon) - f(a)} \frac{f(a+\epsilon) - f(a)}{\epsilon}.$$

Now, the limit of the left hand side as $\epsilon \to 0$ is $h'(a)$; so the limit of the right hand side exists. Since $\lim_{\epsilon \to 0}(f(a+\epsilon) - f(a)) = 0$,

$$\lim_{\epsilon \to 0} \frac{g(f(a+\epsilon)) - g(f(a))}{f(a+\epsilon) - f(a)} = g'(f(a)).$$

Hence we get that

$$\lim_{\epsilon \to 0} \frac{f(a+\epsilon) - f(a)}{\epsilon}$$

exists since $g'(f(a)) \neq 0$, and $f'(a) = (h'(a))(g'(f(a))^{-1}$. Since h and g are analytic and f is continuous, the RHS is continuous, hence f is analytic. □